Introduction to Mathematical Logic Sixth Edition Elliott Mendelson

## 1.4 An Axiom System for the Propositional Calculus

A formal theory $\mathcal{S}$ is defined when the following conditions are satisfied:

1. A countable set of symbols is given as the symbols of $\mathcal{S}$ .* A finite sequence of symbols of $\mathcal{S}$ is called an expression of $\mathcal{S}$.

2. There is a subset of the set of expressions of $\mathcal{S}$ called the set of well-formed formulas (wfs) of $\mathcal{S}$. There is usually an effective procedure to determine whether a given expression is a wf.

3. There is a set of wfs called the set of axioms of $\mathcal{S}$. Most often, one can effectively decide whether a given wf is an axiom; in such a case, $\mathcal{S}$ is called an axiomatic theory.

4. There is a finite set $\mathcal{R}_1, \ldots, \mathcal{R}_n$ of relations among wfs, called rules of inference. For each $\mathcal{R}_i$, there is a unique positive integer j such that, for every set of j wfs and each wf $\mathcal{B}$, one can effectively decide whether the given j wfs are in the relation $\mathcal{R}_i$ to $\mathcal{B}$, and, if so, $\mathcal{B}$ is said to follow from or to be a direct consequence of the given wfs by virtue of $\mathcal{R}_i$.†

A proof in $\mathcal{S}$ is a sequence $\mathcal{B}_1, \ldots, \mathcal{B}_k$ of wfs such that, for each i, either $\mathcal{B}_i$ is an axiom of $\mathcal{S}$ or $\mathcal{B}_i$ is a direct consequence of some of the preceding wfs in the sequence by virtue of one of the rules of inference of $\mathcal{S}$.

A theorem of $\mathcal{S}$ is a wf $\mathcal{B}$ of $\mathcal{S}$ such that $\mathcal{B}$ is the last wf of some proof in $\mathcal{S}$. Such a proof is called a proof of $\mathcal{B}$ in $\mathcal{S}$.

Even if $\mathcal{S}$ is axiomatic—that is, if there is an effective procedure for checking any given wf to see whether it is an axiom—the notion of "theorem" is not necessarily effective since, in general, there is no effective procedure for determining, given any wf $\mathscr{B}$, whether there is a proof of $\mathscr{B}$. A theory for which there is such an effective procedure is said to be decidable; otherwise, the theory is said to be undecidable.

From an intuitive standpoint, a decidable theory is one for which a machine can be devised to test wfs for theoremhood, whereas, for an undecidable theory, ingenuity is required to determine whether wfs are theorems.

A wf $\mathcal{C}$ is said to be a consequence in $\mathcal{S}$ of a set Γ of wfs if and only if there is a sequence $\mathscr{B}_1, \ldots, \mathscr{B}_k$ of wfs such that $\mathcal{C}$ is $\mathscr{B}_k$ and, for each i, either $\mathscr{B}_i$ is an axiom or $\mathscr{B}_i$ is in Γ, or $\mathscr{B}_i$ is a direct consequence by some rule of inference of some of the preceding wfs in the sequence. Such a sequence is called a proof (or deduction) of $\mathcal{C}$ from Γ. The members of Γ are called the hypotheses or premises of the proof. We use $\Gamma \vdash \mathcal{C}$ as an abbreviation for "$\mathcal{C}$ is a consequence of Γ". In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_{\mathcal{S}} \mathcal{C}$, adding the subscript $\mathcal{S}$ to indicate the theory in question.

If Γ is a finite set $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\}$, we write $\mathscr{H}_1, \ldots, \mathscr{H}_m \vdash \mathcal{C}$ instead of $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\} \vdash \mathcal{C}$. If Γ is the empty set ∅, then $\varnothing \vdash \mathcal{C}$ if and only if $\mathcal{C}$ is a theorem. It is customary to omit the sign "∅" and simply write $\vdash \mathcal{C}$. Thus, $\vdash \mathcal{C}$ is another way of asserting that $\mathcal{C}$ is a theorem.

The following are simple properties of the notion of consequence:

1. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \mathcal{C}$, then $\Delta \vdash \mathcal{C}$.
2. $\Gamma \vdash \mathcal{C}$ if and only if there is a finite subset Δ of Γ such that $\Delta \vdash \mathcal{C}$.
3. If $\Delta \vdash \mathcal{C}$, and for each $\mathscr{B}$ in Δ, $\Gamma \vdash \mathscr{B}$, then $\Gamma \vdash \mathcal{C}$.