formula with Gödel number z is not demonstrable'—or, to put it another way, 'No proof can be adduced for the formula with Gödel number z'.

What Gödel showed is that a certain special case of this formula is not formally demonstrable. To construct this special case, we begin with the formula displayed as line (1):

$$\sim(\exists x)\ \mathrm{Dem}\ (x, \mathrm{Sub}\ (y, 17, y)) \qquad (1)$$

This formula belongs to PM, but it possesses a meta-mathematical interpretation. The question is, which one? The reader should recall that the expression 'Sub (y, 17, y)' designates a number. This number is the Gödel number of the formula obtained from the formula with Gödel number y by substituting for the variable with Gödel number 17 (i.e., for all occurrences of the letter 'y') the numeral for y.[31] It will then be evident that the formula on line (1) represents the meta-mathematical statement: 'The formula with Gödel number sub (y, 17, y) is not demonstrable'. Although this is a tantalizing statement, it is still open-ended and indefinite, since it still contains the variable 'y'. To

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[31] It is crucial to recognize that 'Sub (y, 17, y)', though an expression of PM, is not a formula but a name-function for identifying a *number* (see footnote 28). The number so identified will be the Gödel number of a specific formula. Or rather, it would be, were 'y' not a variable. Since 'y' is a variable and not a numeral, the expression 'Sub (y, 17, y)' doesn't represent a specific number any more than does the string 'y + sss0'. For that, the variable 'y' would need to be replaced by a specific numeral.

make it definite, we need a numeral in place of a variable. What numeral should we choose? Here we shall follow Gödel.

Since the formula on line (1) belongs to PM, it has a (very large) Gödel number that could, in principle, be calculated. Luckily, we shall not actually calculate it (nor did Gödel); we shall simply designate its value by the letter '*n*'. We now proceed to replace all occurrences of the variable '*y*' in formula (1) by the number *n* (more precisely, by the *numeral* for the number *n*, which we will blithely write as '*n*', just as we will write '17', knowing that we really mean 'sssssssssssssssss0'). This will yield a new formula, which we shall call 'G':

(G) $\sim(\exists x)$ Dem $(x, \text{Sub}\ (n, 17, n))$

This is the formula we promised. As it is a specialization of the formula on line (1), its meta-mathematical meaning is simply: 'The formula with Gödel number sub (*n*, 17, *n*) is not demonstrable'. And now, as there are no (unquantified) variables left in it, G's meaning is definite.

The formula G occurs within PM, and therefore it must have a Gödel number, *g*. What is the value of *g*? A little reflection shows that *g* = sub (*n*, 17, *n*).[32] To

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[32] Note the key distinction between the number itself and its formal counterpart inside PM. The former is sub (*n*, 17, *n*), with lowercase 's', while the latter is the string we abbreviate as 'Sub (*n*, 17, *n*)', with uppercase 'S'. Otherwise put, 'sub (*n*, 17, *n*)' denotes an actual *quantity*, much as, say, the informal arithmetical expres-

see this, we need but recall that sub (n, 17, n) is the Gödel number of whatever formula results when we substitute n (or rather, its numeral) for the variable with Gödel number 17 (i.e., for 'y') inside the formula whose Gödel number is that same n itself. But the formula G was obtained in precisely that manner! That is, we started with the formula having Gödel number n; then we replaced all copies of 'y' in it by copies of the numeral for n. And so, sub (n, 17, n) is the Gödel number of G.

We must now recall that G is the mirror image *within* PM of the meta-mathematical statement: 'The formula with Gödel number g is not demonstrable'. It follows, then, that G represents, inside PM, the meta-mathematical statement: 'The formula G is not demonstrable'. In a word, the PM formula G can be construed as asserting of itself that it is not a theorem of PM.

(ii) We come to the second step—the proof that G is not, in fact, a theorem of PM. Gödel's argument showing this resembles the development of the Richard Paradox, but steers clear of its fallacious reasoning.[33] The argument is relatively unencumbered. It pro-

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sion '2 × 5' denotes a *quantity* (namely, ten), whereas 'Sub (n, 17, n)' denotes a number-naming *string* inside PM, much like the number-naming string 'ss0 × sssss0'.

[33] It may be useful to make explicit the resemblance as well as the dissimilarity of the present argument to that of the Richard Paradox. The crux is that G is not identical with the meta-mathematical statement with which it is associated, but only *represents* (or mirrors) the latter within PM. In the Richard Paradox, the